Hierarchies of quantum metrology bounds beyond Cramér-Rao

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The cornerstone of modern quantum metrology is the quantum Cramér-Rao bound (QCRB). This bound identifies a lower limit on the variance of unbiased estimators for an unknown parameter that is encoded into a quantum state. Under generic conditions, this bound can be saturated by an optimal estimator and measurement, provided that many repeated measurements on the system are performed.

However, in the presence of small data sets the QCRB typically largely underestimates the error that can actually be achieved in practice. In this talk, we present a family of generalized bounds on the variance of unbiased estimators that are larger than the QCRB when the sample is small and thereby provide a more realistic limit on the achievable precision of a finite-sample quantum measurement.

Generalized bounds are obtained by imposing stricter unbiasedness conditions. These can be formulated in the form of constraints (i) on the average estimator at different test points within the range of possible values, or (ii) on higher-order derivatives of the estimator. Both of these approaches lead to rich hierarchies of bounds that contain the QCRB at the lowest order. All quantum bounds are derived by optimizing classical bounds over all possible quantum measurements. We provide analytical expressions for the explicit construction of the bounds and of the optimal measurements.

The generalized bounds identify threshold behavior at small data sets by revealing tighter precision limits than the QCRB (see Fig. 1). In the large-data limit, the hierarchy of bounds collapses back onto the QCRB.

From the inverse of these bounds we identify quantum information functions that generalize the quantum Fisher information (QFI). The QFI has widespread applications in quantum information theory beyond its immediate scope in quantum metrology. Our results therefore open up a wide spectrum of new quantum information functions whose full potential still remains to be uncovered.

References

[1] M. Gessner and A. Smerzi, *Hierarchies of Frequentist Bounds for Quantum Metrology: From Cramér-Rao to Barankin*, Phys. Rev. Lett. **130**, 260801 (2023).

Figures

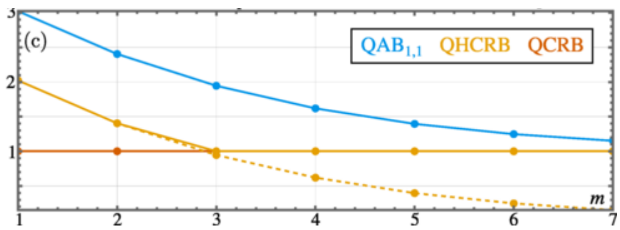

Figure 1: Comparison between the QCRB (orange) and generalized bounds known as the quantum Abel bound (QAB, blue) and the quantum Hammersley-Chapman-Robbins bound (QHCRB, yellow) for a single qubit (see Ref. [1] for details). The plots show the quantum bounds as a function of the number of measurements m (bottom): We observe that for small m, the QCRB is too optimistic.